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Applications of Semigroups

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Absract: - This section deals with the applications of semigroups in general and regular semigroups in particular. The theory of semigroups attracts many algebraists due to their applications to automata theory, formal languages, network analogy etc. In section 2 we have seen different areas of applications of semigroups. We identified some examples in biology, sociology etc. whose semigroup structures are nothing but regular, E-inversive and inverse semigroup etc.

I. INTRODUCTION

The concept of a semigroup is relatively young, the first, often fragmentary, studies were carried out early in the twentieth century. Then the necessity of studying general transformations, rather than only invertible transformations (which played a large role in the development of group theory) became clear. During the past few decades connection in the theory of semigroups and the theory of machines became of increasing importance, both theories enriching each other. In association with the study of machines and automata, other areas of applications such as formal languages and the software use the language of modern algebra in terms of Boolean algebra, semigroups and others. But also parts of other areas, such as biology, psychology, biochemistry and sociology make use of semigroups.

The theory of automata has its origins in the work by Turing (Shannon1948, and Heriken 1994.).Turing developed the theoretical concept of what is now called Turing machines, in order to give computability a more concrete and precise meaning. Hannon investigated the analysis and synthesis of electrical contact circuits using switching algebra. The work of McCullon and pitts centers on neuron models to explain brain functions and neural networks by using finite automata. Their work was continued by Kleene. The development of technology in the areas of electromechanical and machines and particularly computers had a great influence on automata theory which traces back to the mid-1950s. Many different parts of pure mathematicians are used as tools such as abstract algebra, universal algebra, lattice theory, category theory, graph theory, mathematical logic and the theory of algorithms. In turn automata theory can be used in economics, linguistics and learning processes.

The beginning of the study of formal languages can be traced to Chomsky, who introduced the concept of a context-free language in order to model natural languages in 1957. Since then late 1960 there has been considerable activity in the theoretical development of context-free languages both in connection with natural languages and with the programming languages. Chomsky used semi-thue systems to define languages, which can be described as certain subsets of finitely generated free monoids. Chomsky (1957) details a revised approach in the light of experimental evidence and careful consideration of semantic and syntactic structures of sentences. For a common approach to formal languages and the theory of automata we refer to Eilenberg (1974).

Semigroups can be used in biology to describe certain aspects in the crossing of organisms, in genetics and in consideration of metabolisms. The growth of plants can be described algebraically in Hermann and Rosenberg (1975). Further material on this subject is contained in Holcombe (1982). Details on the use of semiautomata in metabolic pathways and the aid of a computer therein, including a theory of scientific experiments, can be found in Krohn, Langer and Rhodes (1976). Rosen (1973) studies ways in which environmental changes can affect the repair capacity of biological systems and considers carcinogenesis and reversibility problems. Language theory is used in cell – development problems, as introduced by Lindenmayes (1968), Hermann and Rosendalg (1975). Suppes (1969) Kiesras (1976) develop a theory of learning in which a subject is instructed to behave like a semiautomaton.

The study of kinship goes back to a study by A. Weil in response to an inquiry by the anthropologist C. Levi-Strauss in 1949. White (1963), Kim and Breiger (1979) and Breiger, Boorman and Srable (1975) are also developed the elementary structure of kinship. Ballonoff (1974) presents several fundamental papers on kinship. Carlso 1980) gives elementary examples of applications of groups in anthropology and sociology. Rudolf Lidl and Guter Pilz were started with a selected set X of basic relations such that the set of all their relation products yields all remaining kinship relations. In this way they arrive at the concept of a free (hence infinite) semigroups over X. Sociology includes the study of human interactive behavior in group situations, in particular, in underlying structures of societies. Such structures can be revealed by mathematical analysis. This indicates how algebraic techniques may be introduced into studies of this kind.

II. SEMIGROUPS AND ITS APPLICATIONS

Now a days the theory of semigroups has been expanded greatly due to its applications to computer science and we also finds its usage in biological science and sociology. In this section we discuss some applications of semigroups in different areas.

2.1. Semigroups –Automaton

The algebraic theory of automata, which uses algebraic concepts to formalize and study certain types of finite-state machines. One of the main algebraic tools used to do this is the theory of semigroups. Automaton is an abstract model of computing device. Using this models different types of problems can be solved. We discuss what is common to all automata by describing an abstract model will be amenable to mathematical treatment and see that there is a close relationship between automata and semigroup. We can establish a correspondence between automata and monoids.

The problem may be identifying or adding two integers etc. i.e., we will be encounting automata in several forms such as calculating machines, computers, money changing devices, telephone switch boards and elevator or left switchings. All the above have one aspect in common namely a "box" which can assume various states. These states can be transformed into other states by outside influence and process "outputs" like results of computations.

Semi automata: A semi automaton is a triple $Y = (Z, A, \delta)$ consisting of two non empty sets Z and A and a function $\delta : Z \times A \rightarrow Z$. Z is called the set of states, A is the set of input alphabet and δ the " next – state function " of Y.

Automata: An automaton is a quintuple $\check{A} = (Z, A, B, \delta, \lambda)$ where (Z, A, δ) is a semi automaton, B is a non empty set called the output alphabet and $\lambda : Z \times A \rightarrow B$ is the " output function."

If $z \in Z$ and $a \in A$ then we interpret $\delta(z, a) \in Z$ as the next state into which z is transformed by the input a. $\lambda(z, a) \in B$ is the output of z resulting from the input a. Thus if the automaton is in the stage z and receives input a, then it changes to state $\delta(z, a)$ with out put $\lambda(z, a)$. A(semi)-automaton is finite, if all the sets Z, A and B are finite, finite automata are also called mealy automata.

Examples: 1) Let $A = \{a_1, a_2, \ldots\ldots a_n\}$, $B = \{b_1, b_2, \ldots\ldots b_m\}$ and $Z = \{z_1, z_2 \ldots\ldots z_k\}$

Description by tables:

Input table

δ	a_1	a_2……………a_n	
z_1	$\delta(z_1, a_1)$	$\delta(z_1, a_2)$	$\delta(z_1, a_1)$
z_2	$\delta(z_2, a_1)$	$\delta(z_2, a_2)$	$\delta(z_2, a_n)$
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
z_k	$\delta(z_k. a_1)$	$\delta(z_k, a_2)$	$\delta(z_k, a_n)$

Out put table

λ	a_1	a_2……………a_n	
z_1	$\lambda(z_1, a_1)$	$\lambda(z_1, a_2)$	$\lambda(z_1, a_1)$
z_2	$\lambda(z_2, a_1)$	$\lambda(z_2, a_2)$	$\lambda(z_2, a_n)$
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
z_k	$\lambda(z_k. a_1)$	$\lambda(z_k, a_2)$	$\lambda(z_k, a_n)$

Description by graphs:

We depict z_1, z_2, ……….z_k as "discs" in the plane and draw an arrow labeled a_i from z_r to z_s, if $\delta(z_r, a_i) = z_s$. In case of an automaton we denote the arrow also by $\lambda(z_r, a_i)$.This graph is called the state graph

Example: (marriage Automation)

Let us consider the following situation in a household . The husband is angry or bored or happy: the wife is quite or shouts or cooks his favorite dish. Silence on her part does not change the husband's mood, shouting "lowers" it by one "degree "(if he is already angry, then no change), cooking of his favorite dish creats general happiness for him. We try to describe this situation in terms of a semi-automaton $Y = (Z, A, \delta)$. We define $Z = (z_1, z_2, z_3)$ and $A = \{a_1, a_2, a_3\}$ with

z_1 = husband is angry a_1 = wife is quite
z_2 = husband is bored a_2 = wife is shout
z_3 = husband is happy a_3 = wife cooks

The following is the description of δ

δ	a_1	a_2	a_3
z_1	z_1	z_1	z_3
z_2	z_2	z_1	z_3
z_3	z_3	z_2	z_3

For this situation add the output $B = \{b_1, b_2\}$ with the interpretation b_1 = husband shouts, b_2 = husband quite. Let us assume that the husband is only shouts if he is angry and his wife shouts. Otherwise he is quite even in state z_3. We shall define the ouput function λ by using the following output table.

λ	a_1	a_2	a_3
z_1	b_2	b_1	b_2
z_2	b_2	b_2	b_2
z_3	b_2	b_2	b_2

Let A^1 be the free monoid on A , i.e., $A^1 = F_A^{(1)}$, the monoid of all finite (including the empty) sequences of elements of the set A. Let us denote the identity (the empty sequence) in A^1 be Λ. We shall extend the next state function δ and the output function λ from $Z \times A$ to $Z \times A^1$ as follows:

For any $z \in Z$ and $a_1 a_2, \ldots an \in A^1$, define

$$\delta^1(z, a_1a_2\ldots a_n) = \begin{cases} z, & \text{if } n = 0 \\ \delta(\delta^1(z, a_1a_2\ldots a_{n-1}), a_n), & \text{if } n > 0. \end{cases}$$

and

$$\lambda^1(z, a_1a_2\ldots a_n) = \begin{cases} \Lambda, & \text{if } n = 0 \\ \lambda(z, a_1)\, \lambda^1(\delta(z, a_1), a_2\ldots a_n), & \text{if } n > 0. \end{cases}$$

In other words, $\delta^1: Z \times A^1 \rightarrow Z$ and $\lambda^1 : Z \times A^1 \rightarrow B^1$ are defined inductively as given above . Note that B^1 is the free monoid on B ; i.e., B^1 is the set of all finite (including the empty) sequences of elements in B.

Now we establish a correspondence between monoids and automata and discuss certain examples. If (S , .) is a monoid and we define $\delta : S \times S \rightarrow S$; $\lambda : S \times S \rightarrow S$ by $\delta(s ,t) = s.t$ and $\lambda(s , t) = s \ \forall s ,t \in S$ .Then (S ,S , S , δ , λ) is an automata and if we define $f_a : S \rightarrow S$ by $f_a (z) = \delta^1 (z ,a)$ for any $a \in A$ and $z \in S$, then { $f_a / a \in A^1$ } is a monoid (under the composition of mappings).

Let us consider the semi automaton given in above marriage automata . First we construct the table for f_Λ, f_{a1}, f_{a2}, f_{a3} indicating their actions on the states z_1, z_2, z_3.

*	f_Λ	f_{a1}	f_{a2}	f_{a3}
$z_1 \rightarrow$	z_1	z_1	z_1	z_3
$z_2 \rightarrow$	z_2	z_2	z_1	z_3
$z_3 \rightarrow$	z_3	z_3	z_2	z_3

Since $f_\Lambda = f_{a1}$, we delete f_{a1} and check whether { f_Λ, f_{a2}, f_{a3} } forms a monoid. The operation "o" is given in the following table.

$\circ$	f_Λ	f_{a2}	f_{a3}
f_Λ	f_Λ	f_{a2}	f_{a3}
f_{a2}	f_{a2}	f_{a2a2}	f_{a3a2}
f_{a3}	f_{a3}	f_{a3}	f_{a3}

$\circ$	F_{a2a2}	f_{a3a2}
$z_1 \rightarrow$	z_1	z_2
$z_2 \rightarrow$	z_1	z_2
$z_3 \rightarrow$	z_1	z_2

Therefore { f_Λ, f_{a2}, f_{a3}} is not a monoid. Now, we extend it to
{f_Λ, f_{a2}, f_{a3}, f_{a3a2}, f_{a3a2}} and verify that this set is a monoid.This is the monoid M corresponding to the automaton A^1.

2.2. Semigroups- Formal languages

A formal language is an abstraction of general characterization of programing language. A grammar is model to describe a language. Given an alphabet or vocabulary (constructing of letters, words, symbols……) we have a method (grammar) for constructing meaningful words or sentences from this alphabets. This immediately reminds us of the term " word semigroup" and indeed these free (word) semigroup well play a major role in constructing language. The formal language L constructed will be a subset of the free semigroup A_* or the free monoid A^* on the alphabet A.

We are familiar with notion of natural languages such as English, Telugu. English language consists of sentences like "Ram walked quickly, Gita ate slowly…". The question is how to define a language, we need some basics. A finite non empty set of symbols denoted by A. Finite sequence of symbols from the alphabet, for example abab, abba are strings over alphabet A = {a, b}.

Concatenation of two strings u and v is the string obtained by appending symbols of v to the right end of u.u = abab, v = abba then uv = abababba. The length of a string u is the number of symbols in the string and denoted by $|u|$. The string with no symbols is denoted by Λ and its length is zero.

If u and v are two strings then $|uv| = |u| + |v|$.

If A is an alphabet, A* is the set of all strings obtained by concatenation zero or more symbols from A, and $A^* = A^+ - \{\Lambda\}$.

If A = {a, b} then A* = {Λ, a, b, ab, aa, ….}.

In general a language is defined as a subset of A^*.

Grammar: A model to describe language is known as grammar. There are essentially three ways to construct a language.

I) Approach via grammar

II) Approach via automata

III) Algebraic approach

Example: Let $A = \{a\}$ and $G = \{g_0\}$, $g_0 \neq a$ and $\rightarrow = \{ g_0 \rightarrow a, g_0 \rightarrow a\, g_0 \}$, $G^1 = \{A, G, \rightarrow, g_0\}$. Then $g_0 = z$ implies $z = a$ or $z = a\, g_0$. Again there is no x with $a \rightarrow x$, $a\, g_0 \rightarrow y$ is only possible for $y = aa$ or $y = aa\, g_0$. Thus $aa\, g_0 \rightarrow aaa$ and $aa\, g_0 \rightarrow aaa\, g_0$ etc.

$$
\begin{array}{c}
g_0 \rightarrow a \\
\downarrow \\
ag_o \rightarrow aa \\
\downarrow \\
aag_o \rightarrow aaa \\
\downarrow \\
aaag_o \rightarrow \ldots.. \\
. \\
. \\
.
\end{array}
$$

Cancellation of elements outside A^* gives us the result L(G) = {a, aa, aaa, …..} = A^*

2.3. Semigroups in Biology

Semigroup can be used in biology to describe certain aspects in the crossing of organisms, in genetics, and in consideration of metabolisms.

Example: In breeding a strain of cattle, which can be block or brown monochromatic or spotted, it is known that black is dominant and brown receive and that monochromatic is dominant over spotted. Thus there are four possible types of cattle in this herd.

a = Black monochromatic c = Brown monochromatic

b = black spotted d = brown spotted.

Due to dominance, in crossing a black spotted one with a brown monochromatic one, we expect a black monochromatic one. This can be symbolized by " b*c = a". The operation '* 'can be studied for all possible pairs to obtain the table.

*	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Then S = {a, b, c, d}is a semigroup with identity element d.

In general, the table for breeding operations is more complicated .

We can ask for connections between hereditary laws and the corresponding semigroups. Of course, we need SoS = S for such semigroups S, since $s \in S \backslash S^2$ would vanish after the first generation and would not even be observed. A different biological problem which leads to algebraic problems is as follows: all genetic information in an organisms is given in the so called deoxy ribonucleic acid(DNA) which consists of two stands which are combined together to form the famous double helix. Each stand is made up as a polymer of four different basics substances, the nucleotides. If the nucleotides are denoted by n_1, n_2, n_3 and n_4, then the stand can be regarded as a word. Over $\{n_1, n_2, n_3, n_4\}$. DNA cannot put the genetic information into effect. By means of a messenger ribonucleic acid, the information contained in the DNA is copied ("translation") and then transferred onto the protein chains ("tranlation"). These protein chains are polymers consisting of 21 different basic substances, the amino acids, denoted by a_1, a_2, a_3, a_4 …..a_{21}. As with the DNA each protein chain can be regarded as a word over $\{a_1, a_2, a_3 \ldots a_{21}\}$.

In general, it is assumed that the sequence of nucleotides in the DNA is uniquely detemined by the sequence of amino acids in a protein chain. In otherwords, it is assumed that there is a monomorphism from the free semigroup $F_{21} = \{a_1, a_2, a_3, \ldots a_{21}\}_*$ into the free semigroup $F_4 = \{n_1, n_2, n_4, n_4\}_*$.

2.4. Semigroups in Sociology

Sociology includes the study of human interactive behavior in group situations, in particular in underlying structures of societies. Such structures can be revealed by mathematical analysis. This indicates how algebraic techniques may be introduced into studies of this kind. So the study of such relations can be elegantly formulated in the language of semigroups.

Rudolf lidl and Gunter Pilz defind relation semigroup as follows.

Definition: (R(M), o) is called the relation semigroup on M. The operation o is called the relation product, where M is a monoid.

It is obvious that $R \in R(M)$ is transitive if and only if $RoR \subseteq R$. The set of their relation products yields all remaining kinship relations. In this way they arrive at the concept of a free semigroup over X. But there are only finite people on the earth and some kinship relations like "daughter of a mother" and " daughter of a father" might be considered to be "the same'.

Definition: A kinship system is a semigroup S = {X, R} where R is a relation on X, which express equality of kinship relationships.

Examples: 1) Let X = {"is father of ", "is mother of "} and R = ϕ. Then the kinship system S is the semigroup {"is father of ", "is mother of", "is grand father on fathers side of "…}.

2) Let F = "is father of", M = " is mother of ", S = " is son of ', D = "is daughter of,' , B = "is brother of", C = "is child of " and Si = "is sister of" then X = {F, M, S, D, B, Si, C} and R = {(CM, CF), (BS, S), (SiD, D), (CBM, CMM), (MC, FM), (SB, S), (Dsi, D), (MBC, MMC)……......}.

For complete list of R see Boyd, Haehl and Sailer (1972) [6]. The first means that in the semigroup we have CM = CF, children of the mother are the same as the children of the father.

Let G be a society i.e, a non empty set of people and let S(G) be the semigroup of all different kinship relationships of this society.

The semigroup S(G) often have special properties, eg. "is son of " and "is father of" are nearly inverse relations. The framework for investigation of special S(G) would be the theory of inverse semigroups, i.e, semigroups S such that for all $s \in S$ with $ss^1s = s$ and $s^1ss^1 = s^1$. For instance, (Mn(R), .) is an inverse semigroup for all $n \in N$. If s denotes "is son of" and s^1 denotes "is father of " then these two equations hold in most societies.

2.5 Further Study

This section deals with the applications of semigroups in general and regular semigroups in particular. The theory of semigroups attracts many algebraists due to their applications to automata theory, formal languages, network analogy etc. In section 2 we have seen different areas of applications of semigroups. We identified some examples in biology, sociology etc. whose semigroup structures are nothing but regular, E-inversive and inverse semigroup etc.

For consider the following examples:

Examples:

1) Let the kinship system S = {X, R} be defined by X = { P = (" is parent of "), C = ("is child of")}, R = {(PP, P), (PC, CP), (CC, C)}. Let a, b, c be the equivalence classes of P, C and PC respectively. Now "o" is given by

·	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

Then (S, .) is a semigroup

2) Let X = {F, M} and R = {(FF, F), (MM, M), (FM, FM)} then
S = {[F], [M], [FM]} with the following table is semigroup.

·	F	M	FM
F	F	FM	FM
M	FM	M	FM
FM	FM	FM	FM

And it is clearly,

FFF = F, MMM = M, FMFMFM = FM, FMFFM = FM, FMMFM = FM

* So, S is a regular semigroup which is not an inverse semigroup.

3) Let X = {F, M} and R = {(FFF, F), (MM, M), (FM, MF)} with the operatable given below is semigroup.

·	F	FF	M	FM	FFM
F	FF	F	FM	FFM	FM
FF	F	FF	F FM	FM	FFM
M	FM	FFM	M	FM	FFM
FM	FFM	FM	FM	FFM	FM
FFM	FM	FFM	FF	FM	FFM

- 4) Let X = {S, M, F} and R = {(SS, S), (MM, M), (FF, F), (SM, S), (SF, S), (MF, M), (MS, M)} then S = {[S], [M], [F]} is an inverse semigroup with the operation table as shown below
-

·	S	M	F
S	S	S	S
M	M	M	M
F	F	F	F

* 5) Let Si = sister and D = daughter then
X = {Si, D} and R = {(SiD, D), (DSi, D), (SiSi, Si), (DD, D)} then S = {[Si], [D]}is regular semigroup but not inverse semigroup with the following operation table.

·	Si	D
Si	Si	D
D	D	Si

6) Let X = {F, M} and R = {(FF, F), (MM, M), (FM, MF)} then S = {[F], [M], [MF], [FM]} with the following operation table is semigroup.

·	F	M	FM	MF
F	F	FM	FM	MF
M	MF	M	FM	MF
FM	MF	FM	FM	MF
MF	MF	MF	FM	MF

Further we want to study some more structures of semigroups which will find applications in different areas.

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